

# Generalised Information Systems Capture L-Domains<sup>\*†‡</sup>

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## Abstract

A generalisation of Scott's information systems [14] is presented that captures exactly all L-domains. The global consistency predicate in Scott's definition is relativised in such a way that there is a consistency predicate for each atomic proposition (token) saying which finite sets of such statements express information that is consistent with the given statement.

It is shown that the states of such generalised information systems form an L-domain, and that each L-domain can be generated in this way, up to isomorphism. Moreover, the equivalence of the category of generalised information systems with the category of L-domains is derived. In addition, it will be seen that from every generalised information system capturing an algebraic bounded-complete domain a corresponding Scott information system can be obtained in an easy and natural way; similarly for Hoofman's continuous information systems [9] and the continuous bounded-complete domains captured by them.

## 1 Introduction

In 1982, in his seminal paper [14], Dana Scott introduced information systems as a logic-based approach to domain theory. An information system consists of a set of tokens to be thought of as atomic statements about a computational process, a consistency predicate telling us which finite sets of such statements contain consistent information, and an entailment relation saying what atomic statements are entailed by which consistent sets of these. Theories of such a logic, also called states, i.e. finitely consistent and entailment-closed sets of atomic statements, form an algebraic bounded-complete domain with respect to set inclusion, and, conversely, every such domain can be obtained in this way, up to isomorphism. This gives Scott's idea that domain elements represent information about states of a computation a precise mathematical meaning.

The role of bounded completeness becomes also clear in this context: States represent consistent information. So, any finite collection of substates must contain consistent information as well, and this fact is witnessed by any of its upper bounds.

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Whereas in Scott’s approach the consistency witnesses are hidden, in this paper we present an approach that makes them explicit. This allows to consider the more general situation in which there is no longer a uniform global consistency predicate. Instead there is a consistency predicate for each atomic statement telling us which finite sets of atomic statements express information that is consistent with the given statement. As it turns out the theories, or states, of such a more general information system form an L-domain, and, up to isomorphism, each L-domain can be obtained in this way.

Since every token in the just delineated kind of information system has its own consistency predicate, we can also think of each such system as a family of logics, or a Kripke frame.

L-domains were independently introduced by Coquand [5] and Jung [10]. As was shown by Jung [10, 11], they form one of the two maximal Cartesian closed full subcategories of the category of continuous domains with Scott continuous functions.

Here, we show that the category of generalized information systems and approximable mappings is equivalent to the category of L-domains. Similar results are derived for algebraic L-domains and bounded-complete domains. In both cases the corresponding generalised information systems satisfy just one additional condition.

Note that a logic-oriented approach to L-domains has also been presented by Zhang [19]. However, the representation considered in that paper is motivated by Gentzen-style proof systems and therefore differs from Scott’s original approach. Moreover, only algebraic L-domains are captured and the function space construction is not considered. Chen and Jung [4] developed a logic for describing algebraic L-domains following Abramsky’s Domain Theory in Logical Form approach [1].

A Scott-style logic-oriented approach capturing general L-domains was introduced by the present author in [15]. However, as in the other approaches, consistency witnesses were hidden. Each consistent set was required to contain its witness, which led to the unsatisfying situation that subsets of consistent sets needed not be consistent again. The problem is eliminated in the present approach. All requirements now have a clear logical meaning.

Scott’s original motivation for the introduction of information systems was to provide a more concrete approach to (abstract) domain theory. Therefore, he presented information system analogues of the domain constructions usually needed in giving a denotational programming language semantics. Especially, the construction of exponents requires special attention in our case. It will be the topic of a following paper [16].

The present paper is organized as follows: Section 2 contains basic definitions and results from domain theory. In Section 3 generalised information systems are considered: the concepts of information frames and information systems with witnesses are introduced and their equivalence is derived. Moreover, it is shown that the states of an information system with witnesses form an L-domain with respect to set inclusion, and that—up to isomorphism—every L-domain can be generated that way. The special cases of algebraic L-domains and bounded-complete domains are considered as well. Approximable mappings between information systems with witnesses are defined in Section 4 and the equivalence between the category of such information systems and mappings and the category of L-domains is shown.

Scott’s information systems are known to represent exactly the algebraic bounded-complete domains. The notion has been generalised by Hoofmann [9] to capture all continuous bounded-complete domains. In Section 5 two classes of information systems with witnesses are considered representing algebraic and continuous bounded-complete domains, respectively, and it is shown that one can pass in an easy and natural way from information systems with witnesses of this kind to Scott and/or Hoofmann information systems in such a way that (up to isomorphism) the same domains are represented.

## 2 Domains: basic definitions and results

For any set  $A$ , we write  $X \subseteq_{\text{fin}} A$  to mean that  $X$  is finite subset of  $A$ . The collection of all subsets of  $A$  will be denoted by  $\mathcal{P}(A)$  and that of all finite subsets by  $\mathcal{P}_f(A)$ .

Let  $(D, \sqsubseteq)$  be a poset.  $D$  is *pointed* if it contains a least element  $\perp$ . For an element  $x \in D$ ,  $\downarrow x$  denotes the principal ideal generated by  $x$ , i.e.,  $\downarrow x = \{y \in D \mid y \sqsubseteq x\}$ . A subset  $S$  of  $D$  is called *consistent* if it has an upper bound.  $S$  is *directed*, if it is nonempty and every pair of elements in  $S$  has an upper bound in  $S$ .  $D$  is a *directed-complete partial order* (dcpo), if every directed subset  $S$  of  $D$  has a least upper bound  $\bigsqcup S$  in  $D$ , and  $D$  is *bounded-complete* if every consistent subset of  $D$  has a least upper bound in  $D$ .

Assume that  $x, y$  are elements of a poset  $D$ . Then  $x$  is said to *approximate*  $y$ , written  $x \ll y$ , if for any directed subset  $S$  of  $D$  the least upper bound of which exists in  $D$ , the relation  $y \sqsubseteq \bigsqcup S$  always implies the existence of some  $u \in S$  with  $x \sqsubseteq u$ . Moreover,  $x$  is *compact* if  $x \ll x$ . A subset  $B$  of  $D$  is a *basis* of  $D$ , if for each  $x \in D$  the set  $\downarrow_B x = \{u \in B \mid u \ll x\}$  contains a directed subset with least upper bound  $x$ . Note that the set of all compact elements of  $D$  is included in every basis of  $D$ . A directed-complete partial order  $D$  is said to be *continuous* (or a *domain*) if it has a basis and it is called *algebraic* (or an *algebraic domain*) if its compact elements form a basis. A pointed bounded-complete domain is called *bc-domain*. Standard references for domain theory and its applications are [8, 7, 2, 17, 3, 6].

**Lemma 2.1** *In a poset  $D$  the following statements hold for all  $x, y, z \in D$ :*

1. *The approximation relation  $\ll$  is transitive.*
2.  *$x \ll y \implies x \sqsubseteq y$ .*
3.  *$x \ll y \sqsubseteq z \implies x \ll z$ .*
4. *If  $D$  has a least element  $\perp$ , then  $\perp \ll x$ .*
5. *If  $F \subseteq \downarrow x \cap \downarrow y$  such that the least upper bounds  $\bigsqcup^x F$  and  $\bigsqcup^y F$ , respectively, exist relative to  $\downarrow x$  and  $\downarrow y$ , then*

$$x, y \sqsubseteq z \implies \bigsqcup^x F = \bigsqcup^y F.$$

6. *If  $D$  is a continuous domain with basis  $B$ , and  $M \subseteq_{\text{fin}} D$ , then*

$$M \ll x \implies (\exists v \in B) M \ll v \ll x,$$

*where  $M \ll x$  means that  $m \ll x$ , for all  $m \in M$ .*

Property 6 is known as the *interpolation law*.

**Definition 2.2** Let  $D$  and  $D'$  be posets. A function  $f: D \rightarrow D'$  is *Scott continuous* if it is monotone and for any directed subset  $S$  of  $D$  with existing least upper bound,

$$\bigsqcup f(S) = f(\bigsqcup S).$$

With respect to the pointwise order the set  $[D \rightarrow D']$  of all Scott continuous functions between two dcpo's  $D$  and  $D'$  is a dcpo again. Observe that it need not be continuous even if  $D$  and  $D'$  are. This is the case, however, if  $D'$  is an L-domain [2].

**Definition 2.3** A pointed<sup>1</sup> domain  $D$  is an  $L$ -domain, if each pair  $x, y \in D$  bounded above by  $z \in D$  has a least upper bound  $x \sqcup^z y$  in  $\downarrow z$ .

Obviously, every bc-domain is an L-domain. As has been shown by Jung [10, 11], the category  $\mathbf{L}$  of L-domains is one of the two maximal Cartesian closed full subcategories of the category  $\mathbf{CONT}_\perp$  of pointed domains and Scott continuous maps. The same holds for the category  $\mathbf{aL}$  of algebraic L-domains with respect to the category  $\mathbf{ALG}_\perp$  of pointed algebraic domains. The one-point domain is the terminal object in these categories and the categorical product  $D \times E$  of two domains  $D$  and  $E$  is the Cartesian product of the underlying sets ordered coordinatewise.

### 3 Generalised information systems

In this section, the ideas outlined in the introduction are made precise: We introduce two—equivalent—generalisations of information systems and study their relationship with L-domains. First, information frames will be considered.

An information frame consists of a Kripke frame  $(A, R)$ , the nodes of which are also called tokens. Associated with each node  $i \in A$  is a consistency predicate  $\text{Con}_i$  classifying the finite sets of tokens which are consistent with respect to node  $i$ , and an entailment relation  $\vdash_i$  between  $i$ -consistent sets and tokens.

The conditions that have to be satisfied are grouped. There are requirements which consistency predicate and entailment relation of each single node have to meet, and which are well known from Scott's information systems. In addition, we find conditions that specify their interplay for nodes related to each other by the accessibility relation.

**Definition 3.1** Let  $A$  be a set,  $R$  be a binary relation on  $A$ ,  $\Delta \in A$ ,  $(\text{Con}_i)_{i \in A}$  be a family of subsets of  $\mathcal{P}_f(A)$ , and  $(\vdash_i)_{i \in A}$  be a family of relations  $\vdash_i \subseteq \text{Con}_i \times A$ . Then  $(A, R, (\text{Con}_i)_{i \in A}, (\vdash_i)_{i \in A}, \Delta)$  is an *information frame* if the following conditions hold, for all  $i, j, a \in A$  and all finite subsets  $X, Y$  of  $A$ :

1.  $\{i\} \in \text{Con}_i$
2.  $Y \subseteq X \wedge X \in \text{Con}_i \Rightarrow Y \in \text{Con}_i$
3.  $\emptyset \vdash_i \Delta$

and, defining  $X \vdash_i Y$  to mean that  $X \vdash_i b$ , for all  $b \in Y$ ,

4.  $X \in \text{Con}_i \wedge X \vdash_i Y \Rightarrow Y \in \text{Con}_i$
5.  $X, Y \in \text{Con}_i \wedge Y \supseteq X \wedge X \vdash_i a \Rightarrow Y \vdash_i a$
6.  $X \in \text{Con}_i \wedge X \vdash_i Y \wedge Y \vdash_i a \Rightarrow X \vdash_i a$
7.  $X \in \text{Con}_i \wedge X \vdash_i a \Rightarrow (\exists Z \in \text{Con}_i) X \vdash_i Z \wedge Z \vdash_i a$
8.  $iRj \Rightarrow \text{Con}_i \subseteq \text{Con}_j$
9.  $\{i\} \in \text{Con}_j \Rightarrow iRj$ .
10.  $iRj \wedge X \in \text{Con}_i \wedge X \vdash_i a \Rightarrow X \vdash_j a$

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<sup>1</sup>Note that in [6] pointedness is not required.

11.  $iRj \wedge X \in \text{Con}_i \wedge X \vdash_j a \Rightarrow X \vdash_i a$
12.  $X \vdash_i Y \Rightarrow (\exists e \in A) X \vdash_i e \wedge Y \in \text{Con}_e$ .

All requirements are very natural: Each token witnesses its own consistency (1). If the consistency of some set is witnessed by  $i$ , the same holds for all of its subsets (2).  $\Delta$  is entailed by any set of information and in every node, i.e., it represents global truth (3). By (4) each entailment relation preserves consistency. If a set  $X$  entails  $a$ , so does any bigger set (5). Entailment is idempotent (6, 7). In particular it is transitive. Consistency and entailment are preserved when moving from a node  $i$  to its accessible neighbour  $j$  (8, 10). Moreover, entailment is conservative: what is  $j$ -entailed from an  $i$ -consistent set is already  $i$ -entailed (11). Condition (12), finally, states an interpolation property strengthening (4).

**Lemma 3.2** *Let  $A$  be a set,  $R$  be a binary relation on  $A$ ,  $\Delta \in A$ ,  $(\text{Con}_i)_{i \in A}$  be a family of subsets of  $\mathcal{P}_f(A)$ , and  $(\vdash_i)_{i \in A}$  be a family of relations  $\vdash_i \subseteq \text{Con}_i \times A$  such that Axioms 3.1(8, 9) hold. Then the following two statements hold:*

1. *If Axiom 3.1(4) is satisfied, then for all  $i, j \in A$  and all  $X \in \text{Con}_i$ ,*

$$X \vdash_i j \Rightarrow jRi. \quad (1)$$

2. *If Condition (1) and Axiom 3.1(12) are satisfied, so is Axiom 3.1(4).*

**Proof:** (1) If  $X \vdash_i j$ , then  $\{j\} \in \text{Con}_i$ , by Axiom 3.1(4), and thus  $jRi$ , because of Condition 3.1(9).

(2) Assume that  $X \vdash_i Y$ . Because of Axiom 3.1(12) there is some  $j \in A$  such that  $X \vdash_i j$  and  $Y \in \text{Con}_j$ . With (1) we obtain that  $jRi$  and hence with Condition 3.1(8) that  $Y \in \text{Con}_i$ .

By Axiom 3.1(7) the entailment relation of each node of an information frame satisfies an interpolation condition. As we will see now, the frame also has a global interpolation property.

**Lemma 3.3** *Let  $A$  be a set,  $R$  be a binary relation on  $A$ ,  $\Delta \in A$ ,  $(\text{Con}_i)_{i \in A}$  be a family of subsets of  $\mathcal{P}_f(A)$ , and  $(\vdash_i)_{i \in A}$  be a family of relations  $\vdash_i \subseteq \text{Con}_i \times A$  such that Axioms 3.1(4, 5, 8-11) are satisfied. Then Axioms 3.1(7, 12) hold if, and only if, for all  $i \in A$ ,  $X \in \text{Con}_i$  and  $F \subseteq_{\text{fin}} A$ ,*

$$X \vdash_i F \Rightarrow (\exists j \in A)(\exists Y \in \text{Con}_j) X \vdash_i j \wedge X \vdash_i Y \wedge Y \vdash_j F.$$

**Proof:** For the “only-if”-part assume that  $X \vdash_i F$  and  $a \in F$ . By Axiom 3.1(7) there is some  $Y_a \in \text{Con}_i$  with  $X \vdash_i Y_a$  and  $Y_a \vdash_i a$ . Set  $Y = \bigcup \{Y_a \mid a \in F\}$ . Then  $X \vdash_i Y$ . Thus,  $Y \in \text{Con}_i$ , by Axiom 3.1(4). Because of Condition 3.1(12) there is some  $j \in A$  with  $X \vdash_i j$  and  $Y \in \text{Con}_j$ . With Lemma 3.2(1) it follows that  $jRi$ . Since moreover  $Y \vdash_i F$ , because of Requirement 3.1(5), we obtain with Axioms 3.1(8, 11) that  $Y \vdash_j F$ .

Let us next prove the “if”-direction. We first verify Axiom 3.1(7). Suppose that  $X \in \text{Con}_i$  with  $X \vdash_i a$ . Then there are  $e \in A$  and  $Z \in \text{Con}_e$  so that  $X \vdash_i e$ ,  $X \vdash_i Z$  and  $Z \vdash_e a$ . With Lemma 3.2(1) it follows that  $eRi$ . Hence  $Z \in \text{Con}_i$ , by Axiom 3.1(8). Moreover,  $Z \vdash_i a$ , because of Condition 3.1(10).

It remains to derive Axiom 3.1(12). Assume that  $X \vdash_i Y$ . Then there are  $e \in A$  and  $Z \in \text{Con}_e$  with  $X \vdash_i e$  and  $Z \vdash_e Y$ , the latter implying that  $Y \in \text{Con}_e$ , because of Axiom 3.1(4).

As a consequence of Axioms 3.1(1, 8, 9), we have that

$$iRj \iff \{i\} \in \text{Con}_j.$$

It follows that  $R$  is a preorder. Moreover, it is uniquely determined by the consistency predicates and can thus be omitted from the definition of an information frame.

Now, set

$$\text{CON} = \{ (i, X) \mid i \in A \wedge X \in \text{Con}_i \}$$

and for  $(i, X) \in \text{CON}$  and  $a \in A$ ,

$$(i, X) \vdash a \iff X \vdash_i a.$$

Then  $\vdash \subseteq \text{CON} \times A$  and  $\text{CON} \subseteq A \times \mathcal{P}_f(A)$  such that

$$\text{CON}(i) = \{ X \mid (i, X) \in \text{CON} \} = \text{Con}_i.$$

This shows that information frames can also be written in an information systems-like style with a global entailment relation and a consistency predicate that forces consistent sets to explicitly show their consistency witness, and vice versa.

**Definition 3.4** Let  $A$  be a set,  $\Delta \in A$ ,  $\text{CON} \subseteq A \times \mathcal{P}_f(A)$ , and  $\vdash \subseteq \text{CON} \times A$ . Then  $(A, \text{CON}, \vdash, \Delta)$  is an *information system with witnesses* if the following conditions hold, for all  $i, j, a \in A$  and all finite subsets  $X, Y$  of  $A$ :

1.  $\{i\} \in \text{CON}(i)$
2.  $Y \subseteq X \wedge X \in \text{CON}(i) \Rightarrow Y \in \text{CON}(i)$
3.  $(i, \emptyset) \vdash \Delta$
4.  $X \in \text{CON}(i) \wedge (i, X) \vdash Y \Rightarrow Y \in \text{CON}(i)$
5.  $X, Y \in \text{CON}(i) \wedge X \subseteq Y \wedge (i, X) \vdash a \Rightarrow (i, Y) \vdash a$
6.  $X \in \text{CON}(i) \wedge (i, X) \vdash Y \wedge (i, Y) \vdash a \Rightarrow (i, X) \vdash a$
7.  $\{i\} \in \text{CON}(j) \Rightarrow \text{CON}(i) \subseteq \text{CON}(j)$
8.  $\{i\} \in \text{CON}(j) \wedge X \in \text{CON}(i) \wedge (i, X) \vdash a \Rightarrow (j, X) \vdash a$
9.  $\{i\} \in \text{CON}(j) \wedge X \in \text{CON}(i) \wedge (j, X) \vdash a \Rightarrow (i, X) \vdash a$
10.  $(i, X) \vdash Y \Rightarrow (\exists (e, Z) \in \text{CON})(i, X) \vdash (e, Z) \wedge (e, Z) \vdash Y$

where  $(i, X) \vdash (e, Z)$  means that  $(i, X) \vdash e$  and  $(i, X) \vdash Z$ .

Sometimes a stronger axiom than (6) is needed which reverses Axiom (10).

**Lemma 3.5** Let  $(A, \text{CON}, \vdash, \Delta)$  be an information system with witnesses. Then the following rule holds, for all  $a \in A$  and  $(i, X), (j, Y) \in \text{CON}$ ,

$$(i, X) \vdash (j, Y) \wedge (j, Y) \vdash a \Rightarrow (i, X) \vdash a.$$

**Proof:** Since  $(i, X) \vdash j$ , it follows with Axiom 3.4(4) that  $\{j\} \in \text{CON}(i)$ . As a consequence of Axioms 3.4(7, 8) we therefore have that  $(i, Y) \vdash a$ . Now, we can apply Axiom 3.4(6) to obtain that  $(i, X) \vdash a$ .

As we have seen, information frames and information systems with witnesses can be derived from each other. Moreover, as will become clear next, the states associated with an information frame are the same as the states associated with the information system with witnesses generated by it, and conversely. In this sense both concepts are equivalent.

**Definition 3.6** Let  $(A, \text{CON}, \vdash, \Delta)$  be an information system with witnesses. A subset  $x$  of  $A$  is a *state* of  $(A, \text{CON}, \vdash, \Delta)$  if the following three conditions hold:

1.  $(\forall F \subseteq_{\text{fin}} x)(\exists i \in x) F \in \text{CON}(i)$
2.  $(\forall i \in x)(\forall X \subseteq_{\text{fin}} x)(\forall a \in A)[X \in \text{CON}(i) \wedge (i, X) \vdash a \Rightarrow a \in x]$
3.  $(\forall a \in x)(\exists i \in x)(\exists X \subseteq_{\text{fin}} x) X \in \text{CON}(i) \wedge (i, X) \vdash a$ .

Using the above relationship between information frames and systems this definition can be rewritten into a corresponding definition for information frames so that related frames and systems induce the same set of states.

As follows from the definition, states are subsets of tokens that are *finitely consistent* (1) and *closed under entailment* (2). Furthermore, each token in a state is *derivable* (3), i.e. for each token the state contains a consistent set and its witness entailing the token.

By Condition 3.6(1) states are never empty: Choose  $F$  to be the empty set. Then the state contains some  $i$  with  $\emptyset \in \text{CON}(i)$ .

Note that Conditions (1, 3) in Definition 3.6 can be replaced by a single requirement.

**Proposition 3.7** Let  $(A, \text{CON}, \vdash, \Delta)$  be an information system with witnesses and  $x$  be a subset of  $A$ . Then Conditions 3.6(1) and (3) are equivalent to the following statement:

$$(\forall F \subseteq_{\text{fin}} x)(\exists i \in x)(\exists X \subseteq_{\text{fin}} x) X \in \text{CON}(i) \wedge (i, X) \vdash F. \quad (\text{ST})$$

**Proof:** For the “if”-part let  $F \subseteq_{\text{fin}} x$  and  $a \in F$ . By Condition 3.6(3) there exist  $i_a \in x$  and  $X_a \subseteq_{\text{fin}} x$  so that  $X_a \in \text{CON}(i_a)$  and  $(i_a, X_a) \vdash a$ . Let  $G = \{i_a \mid a \in F\}$  and  $X = \bigcup \{X_a \mid a \in F\}$ . Then  $G \cup X \subseteq_{\text{fin}} x$ . Hence, by Condition 3.6(1), there is some  $j \in x$  such that  $G \cup X \in \text{CON}(j)$ . With Axiom 3.4(2) we obtain that both  $X_a \in \text{CON}(j)$  and  $\{i_a\} \in \text{CON}(j)$ , for all  $a \in F$ , from which it follows by Axioms 3.4(8) and (5) that  $(j, X) \vdash a$ . Thus,  $(j, X) \vdash F$ .

For the “only-if”-part we only have to show that Condition 3.6(1) holds, the other one being a special case of our assumption. Let  $F \subseteq_{\text{fin}} x$  again. By assumption there are  $i \in x$  and  $X \in \text{CON}(i)$  with  $(i, X) \vdash F$ . Hence  $F \in \text{CON}(i)$ , by Axiom 3.4(4).

With respect to set inclusion the states of  $A$  form a partially ordered set, denoted by  $|A|$ .

**Lemma 3.8**  $|A|$  is directed-complete.

**Proof:** Let  $\mathcal{S}$  be a directed subset of  $|A|$ . It suffices to show that  $\bigcup \mathcal{S}$  is a state as well:

To this end, we first verify Condition (ST) in Proposition 3.7. Let  $F \subseteq_{\text{fin}} \bigcup \mathcal{S}$ . Since  $\mathcal{S}$  is directed, there some  $x \in \mathcal{S}$  with  $F \subseteq x$ . Thus, there are  $i \in x$  and  $X \subseteq_{\text{fin}} x$  with  $X \in \text{CON}(i)$  and  $(i, X) \vdash F$ . Since,  $x \subseteq \bigcup \mathcal{S}$ , we are done.

It remains to show that also Condition 3.6(2) holds. Let  $i \in \bigcup \mathcal{S}$ ,  $X \subseteq_{\text{fin}} \bigcup \mathcal{S}$  and  $a \in A$  such that  $X \in \text{CON}(i)$  with  $(i, X) \vdash a$ . Again, as  $\mathcal{S}$  is directed, there is some  $x \in \mathcal{S}$  with  $i \in x$  and  $X \subseteq x$ . By Condition 3.6(2) we therefore obtain that  $a \in x$ . Hence,  $a \in \bigcup \mathcal{S}$ .

As we will see next, the consistent subsets of  $A$  generate a canonical basis of  $|A|$ . For  $(i, X) \in \text{CON}$  let

$$[X]_i = \{a \in A \mid (i, X) \vdash a\}.$$

**Lemma 3.9** 1.  $[X]_i$  is a state of  $A$ , for each  $(i, X) \in \text{CON}$ .

2. For every  $z \in |A|$ , the set of all  $[X]_i$  with  $\{i\} \cup X \subseteq z$  is directed and  $z$  is its union.

**Proof:** (1) Conditions (ST) and 3.6(2), respectively, are immediate consequences of Axiom 3.4(10) and Lemma 3.5.

For (2) let  $\mathcal{A}_z = \{[X]_i \mid \{i\} \cup X \subseteq z \wedge (i, X) \in \text{CON}\}$ . As  $z$  is a state, there is some  $j \in z$  such that  $(j, \emptyset) \in \text{CON}$ . Thus,  $\mathcal{A}_z$  is not empty. As a further consequence of 3.6(1) we have for  $(i, X), (j, Y) \in \text{CON}$  with  $\{i, j\} \cup X \cup Y \subseteq z$  that there is some  $k \in z$  so that  $(k, \{i, j\} \cup X \cup Y) \in \text{CON}$ . Because of 3.4(2) it follows that  $\{i\}, \{j\}, X, Y, X \cup Y \in \text{CON}(k)$ . With 3.4(8, 5) we therefore obtain that  $[X]_i, [Y]_j \subseteq [X \cup Y]_k$ . Thus,  $\mathcal{A}_z$  is directed. Obviously,  $\bigcup \mathcal{A}_z \subseteq z$ . Conversely, let  $a \in z$ . By applying 3.6(3) we gain  $i \in z$  and  $X \subseteq_{\text{fin}} z$  so that  $(i, X) \vdash a$ . Then  $[X]_i \in \mathcal{A}_z$  and hence  $a \in \bigcup \mathcal{A}_z$ .

This result allows characterizing the approximation relation on  $A$  in terms of the entailment relation. The characterization nicely reflects the intuition that  $x \ll y$  if  $x$  is covered by a “finite part” of  $y$ .

**Proposition 3.10** For  $x, y \in |A|$ ,

$$x \ll y \iff (\exists (i, V) \in \text{CON}) \{i\} \cup V \subseteq y \wedge (i, V) \vdash x.$$

**Proof:** The “if”-part is an obvious consequence of the preceding lemma.

For the proof of the converse implication assume that  $\mathcal{S}$  is a directed collection of states of  $A$  such that  $y \subseteq \bigcup \mathcal{S}$ . By the premise there is some finite subset  $V$  of  $y$  with consistency witness  $i \in y$  such that  $x \subseteq [V]_i$ . It follows that  $\{i\} \cup V \subseteq \bigcup \mathcal{S}$ . Since  $V$  is finite and  $\mathcal{S}$  directed, there is some  $s \in \mathcal{S}$  with  $\{i\} \cup V \subseteq s$ . As  $s$  is a state, we obtain that also  $[V]_i \subseteq s$  and hence that  $x \subseteq s$ . Thus,  $x \ll y$ .

Because of Axioms 3.4(1, 2) we have that  $\emptyset \in \text{CON}(i)$ , for all  $i \in A$ . Moreover, with Axioms 3.4(3, 4), we obtain that  $\{\Delta\} \in \text{CON}(j)$ , also for all  $j \in A$ .

**Lemma 3.11** 1.  $[\emptyset]_i = [\Delta]_j$ , for all  $i, j \in A$ .

2.  $[\emptyset]_\Delta \subseteq x$ , for all  $x \in |A|$ .

**Proof:** (1) With 3.4(6, 3) we have that  $[\Delta]_i \subseteq [\emptyset]_i$ . The converse inclusion follows with 3.4(5). In addition, by applying 3.4(8, 9), we gain that  $[\emptyset]_\Delta = [\emptyset]_i$ . The statement is now an easy consequence.

(2) As states are nonempty, there is some  $i \in x$ . Moreover, by Axiom 3.4(3),  $(i, \emptyset) \vdash \Delta$ . Thus,  $\Delta \in x$ , because of 3.6(2). Hence, by applying the same rule again, we obtain that  $[\emptyset]_\Delta \subseteq x$ .

**Lemma 3.12** Let  $x, y, z \in |A|$  so that  $x, y \subseteq z$ . Then

$$\bigcup \{[Z]_k \mid (k, Z) \in \text{CON} \wedge k \in z \wedge Z \subseteq_{\text{fin}} x \cup y\}$$

is the least upper bound of  $x$  and  $y$  in  $\downarrow z$ .



**Proof:** As in the proof of Lemma 3.9 it follows that  $\{[Z]_k \mid k \in z \wedge Z \subseteq_{\text{fin}} x \cup y\}$  is directed. Thus,  $\bigcup \{[Z]_k \mid k \in z \wedge Z \subseteq_{\text{fin}} x \cup y\} \in |A|$ .

Let  $(i, X) \in \text{CON}$  such that  $\{i\} \cup X \subseteq x$ . Then  $i \in z$  and  $X \subseteq x \cup y$ . Thus,  $[X]_i \in \{[Z]_k \mid k \in z \wedge Z \subseteq_{\text{fin}} x \cup y\}$ . With Lemma 3.9(2) it follows that  $x \subseteq \bigcup \{[Z]_i \mid i \in z \wedge Z \subseteq_{\text{fin}} x \cup y\}$ . In the same way we obtain that  $y \subseteq \bigcup \{[Z]_i \mid i \in z \wedge Z \subseteq_{\text{fin}} x \cup y\}$ .

Finally, let  $u \in |A|$  with  $x, y \subseteq u \subseteq z$ . Moreover, let  $(k, Z) \in \text{CON}$  so that  $k \in z$  and  $Z \subseteq_{\text{fin}} x \cup y$ . Then  $Z \subseteq_{\text{fin}} u$ . Hence, there exist  $e \in u$  such that  $Z \in \text{CON}(e)$ . Since  $\{e, k\} \subseteq z$ , it follows with Property (ST) that there is some  $(d, U) \in \text{CON}$  with  $\{d\} \cup U \subseteq z$  and  $(d, U) \vdash \{e, k\}$ . With 3.4(3, 7-9) we now obtain that  $Z \in \text{CON}(d)$  and  $[Z]_k = [Z]_d = [Z]_e$ . Thus,  $[Z]_k \subseteq u$ .

Let us now sum up what we have shown so far.

**Theorem 3.13** *Let  $(A, \text{CON}, \vdash, \Delta)$  be an information system with witnesses. Then  $\mathcal{L}(A) = (|A|, \subseteq, [\emptyset]_\Delta)$  is an L-domain with basis  $\widehat{\text{CON}} = \{[X]_i \mid (i, X) \in \text{CON}\}$ .*

Next, we study when  $\mathcal{L}(A)$  is algebraic. As a consequence of Lemma 3.5 and Proposition 3.10 we obtain:

**Lemma 3.14** *For  $(i, Z) \in \text{CON}$  the following two statements are equivalent:*

1.  $[Z]_i \ll [Z]_i$
2.  $(\exists(j, V) \in \text{CON})(i, Z) \vdash (j, V) \wedge (j, V) \vdash (j, V) \wedge (j, V) \vdash [Z]_i$ .

**Definition 3.15** Let  $(A, \text{CON}, \vdash, \Delta)$  be an information system with witnesses. An element  $(j, V) \in \text{CON}$  is called *reflexive* if  $(j, V) \vdash (j, V)$ .

Let  $\text{CON}_{\text{refl}}$  denote the subset of reflexive elements of  $\text{CON}$ . Obviously,  $[V]_j$  is compact, for every  $(j, V) \in \text{CON}_{\text{refl}}$ .

**Lemma 3.16** *The following two statements are equivalent:*

1. *For every  $z \in |A|$ , the set of all  $[V]_j$  with  $(j, V) \in \text{CON}_{\text{refl}}$  and  $\{j\} \cup V \subseteq z$  is directed and its union is  $z$ .*
2. *The information system  $A$  satisfies Condition ALG saying that for all  $(i, X) \in \text{CON}$  and  $F \subseteq_{\text{fin}} A$ ,*

$$(i, X) \vdash F \Rightarrow (\exists(j, V) \in \text{CON}_{\text{refl}})(i, X) \vdash (j, V) \wedge (j, V) \vdash F. \quad (\text{ALG})$$

**Proof:** The “if”-part is obvious. For the “only if”-part let

$$\mathcal{B}_z = \{[V]_j \mid (j, V) \in \text{CON}_{\text{refl}} \wedge \{j\} \cup V \subseteq z\}.$$

Then it follows as in the proof of Lemma 3.9(2) that  $\mathcal{B}_z$  is not empty. Let  $(i, U), (j, V) \in \text{CON}_{\text{refl}}$  with  $\{i, j\} \cup U \cup V \subseteq z$ . Then it follows with Property (ST) that there is some  $(k, X) \in \text{CON}$  with  $\{k\} \cup X \subseteq z$  such that  $(k, X) \vdash \{i, j\} \cup U \cup V$ . By Condition (ALG), we furthermore obtain some  $(e, Z) \in \text{CON}_{\text{refl}}$  so that  $(k, X) \vdash (e, Z)$  and  $(e, Z) \vdash \{i, j\} \cup U \cup V$ . Thus,  $[U]_i, [V]_j \subseteq [Z]_e \subseteq [X]_k \subseteq z$ , which shows that  $\mathcal{B}_z$  is directed.

It remains to show that  $z \subseteq \bigcup \mathcal{B}_z$ . Let to this end  $a \in z$ . Because of 3.6(3) there is some  $(k, X) \in \text{CON}$  with  $\{k\} \cup X \subseteq z$  and  $(k, X) \vdash a$ . As we have just seen, by Condition (ALG) there exists  $(j, Y) \in \text{CON}_{\text{refl}}$  such that  $\{j\} \cup Y \in [X]_k \subseteq z$  and  $a \in [Z]_j$ , which means that  $a \in \bigcup \mathcal{B}_z$ .

According to Lemma 3.3, in the presence of Axioms 3.4(4, 5, 7, 8, 9) Condition 3.4(10) holds, exactly if for all  $(i, X) \in \text{CON}$ ,  $a \in A$  and  $F \subseteq_{\text{fin}} A$  the following two requirements hold:

$$(i, X) \vdash a \Rightarrow (\exists Z \in \text{CON}(i))(i, X) \vdash Z \wedge (i, Z) \vdash a \quad (2)$$

$$(i, X) \vdash F \Rightarrow (\exists j \in A)(i, X) \vdash j \wedge F \in \text{CON}(j). \quad (3)$$

This allows to simplify Condition (ALG).

**Lemma 3.17** *Condition (ALG) holds if, and only if, the following Condition (SALG) is satisfied for all  $(i, X) \in \text{CON}$  and  $a \in A$ ,*

$$(i, X) \vdash a \Rightarrow (\exists Z \in \text{CON}(i))(i, X) \vdash Z \wedge (i, Z) \vdash Z \wedge (i, Z) \vdash a. \quad (\text{SALG})$$

**Proof:** Assume first that (ALG) holds, and let  $(i, X) \in \text{CON}$  and  $a \in A$  with  $(i, X) \vdash a$ . Then there is some reflexive  $(j, Z) \in \text{CON}$  such that  $(i, X) \vdash (j, Z)$  and  $(j, Z) \vdash a$ . In particular, we obtain that  $\{j\} \in \text{CON}(i)$  and hence that  $(i, Z) \vdash Z$  as well as  $(i, Z) \vdash a$ .

Next, suppose that (SALG) is satisfied, and let  $(i, X) \in \text{CON}$  and  $F \subseteq_{\text{fin}} A$  with  $(i, X) \vdash F$ . Then, for every  $a \in F$ , there is some  $Z_a \in \text{CON}(i)$  so that  $(i, X) \vdash Z_a$ ,  $(i, Z_a) \vdash Z_a$  and  $(i, Z_a) \vdash a$ . For  $Z = \bigcup_{a \in F} Z_a$  it follows that  $(i, X) \vdash Z$ , from which we obtain that  $Z \in \text{CON}(i)$ . Moreover, we have that  $(i, Z) \vdash Z$  and  $(i, Z) \vdash F$ . Because of Condition (2) there is now some  $j \in A$  with  $(i, Z) \vdash j$  and  $Z \in \text{CON}(j)$ . With Axioms 3.4(6) and 3.4(9) it follows that  $(i, X) \vdash j$ ,  $(j, Z) \vdash (j, Z)$  and  $(j, Z) \vdash F$ , as was to be shown.

**Theorem 3.18** *Let  $(A, \text{CON}, \vdash, \Delta)$  be an information system with witnesses. Then  $\mathcal{L}(A)$  is algebraic if, and only if, information system  $A$  satisfies Condition (ALG).*

As well, we will provide a condition which guaranties that  $\mathcal{L}(A)$  is a bounded-complete.

**Theorem 3.19** *Let  $(A, \text{CON}, \vdash, \Delta)$  be an information system with witnesses. Then  $\mathcal{L}(A)$  is bounded-complete, and hence a bc-domain, if information system  $A$  satisfies Condition (BC) saying that for all  $X \subseteq_{\text{fin}} A$  and  $i, j \in A$ ,*

$$(i, X), (j, X) \in \text{CON} \Rightarrow (\forall a \in A)[(i, X) \vdash a \Leftrightarrow (j, X) \vdash a]. \quad (\text{BC})$$

**Proof:** Since  $|A|$  is directed-complete, it suffices to show that any pair of elements that is bounded above has a least upper bound. Let to this end  $x, y \subseteq z, z'$ . We will prove that  $x \sqcup^z y = x \sqcup^{z'} y$ .

Let  $a \in x \sqcup^z y$ . Then there are  $(k, Z) \in \text{CON}$  such that  $k \in z$ ,  $Z \subseteq x \cup y$ , and  $a \in [Z]_k$ . It follows that  $Z \subseteq_{\text{fin}} z'$ . Thus, because of 3.6(1),  $Z \in \text{CON}(k')$ , for some  $k' \in z'$ . As  $[Z]_k = [Z]_{k'}$ , by Condition (BC), we obtain that  $a \in x \sqcup^{z'} y$ .

It is unknown whether the requirement on  $A$  is also necessary. In what follows, however, we will conversely show that every L-domain  $D$  defines an information system with witnesses the associated L-domain of which is isomorphic to  $D$ . In case that  $D$  is bounded-complete this information system will satisfy Condition (BC).

Let  $(D, \sqsubseteq)$  be an L-domain with basis  $B$  and least element  $\perp$ . Set

$$\mathcal{I}(D) = (B, \text{CON}, \vdash, \perp)$$

with

$$\text{CON} = \{ (i, X) \mid i \in B \wedge X \subseteq_{\text{fin}} \downarrow i \cap B \}$$

and

$$(i, X) \vdash a \iff a \ll \bigsqcup^i X.$$

**Lemma 3.20**  $\mathcal{I}(D)$  is an information system with witnesses.

**Proof:** All conditions in Definition 3.4 are easy consequences of Lemma 2.1. In particular, Condition 10 follows from the interpolation law.

**Lemma 3.21** Every state of  $\mathcal{I}(D)$  is a directed subset of  $D$ .

**Proof:** As we have already seen, states are not empty. Let  $x \in |\mathcal{I}(D)|$  and  $a, b \in x$ . Then it follows with 3.6(1) that  $\{a, b\} \in \text{CON}(i)$ , for some  $i \in x$ . Thus  $a, b \sqsubseteq i$ .

It follows that  $\bigsqcup x$  exists in  $D$ , for every  $x \in |\mathcal{I}(D)|$ . For  $x \in |\mathcal{I}(D)|$  set

$$\text{sp}_D(x) = \bigsqcup x.$$

Then  $\text{sp}_D: |\mathcal{I}(D)| \rightarrow D$  is Scott continuous.

**Lemma 3.22** For  $\alpha \in D$ ,  $\{a \in B \mid a \ll \alpha\}$  is a state of  $\mathcal{I}(D)$ .

**Proof:** Condition (ST) is an immediate consequence of the interpolation law and Condition 3.6(2) is obvious.

Set

$$\text{st}_D(\alpha) = \{a \in B \mid a \ll \alpha\},$$

for  $\alpha \in D$ . Then  $\text{st}_D: D \rightarrow |\mathcal{I}(D)|$  is Scott continuous as well. Since  $B$  is a basis of  $D$ , we have that  $\text{sp}_D(\text{st}_D(\alpha)) = \alpha$ .

**Lemma 3.23** For  $x \in |\mathcal{I}(D)|$ ,  $\text{st}_D(\text{sp}_D(x)) = x$ .

**Proof:** We have that

$$\text{st}_D(\text{sp}_D(x)) = \{a \in B \mid a \ll \bigsqcup x\} = \{a \in B \mid (\exists b \in x) a \ll b\}.$$

If  $a \ll b$ , for some  $b \in x$ , we obtain by 3.6(3) that there is some  $(i, X) \in \text{CON}$  with  $\{i\} \cup X \subseteq x$  such that  $a \ll b \ll \bigsqcup^i X$ , from which it follows that  $a \ll \bigsqcup^i X$ . Hence,  $(i, X) \vdash a$ . By 3.6(2) we gain that  $a \in x$ .

Conversely, if  $a \in x$ , then, again by 3.6(3), there is some  $(i, X) \in \text{CON}$  so that  $\{i\} \cup X \subseteq x$  and  $(i, X) \vdash a$ . It follows that  $a \ll \bigsqcup^i X \subseteq i$ , and hence that  $a \ll i$ .

Thus, both functions are inverse to each other, which shows that  $D$  is isomorphic to  $|\mathcal{I}(D)|$ .

**Theorem 3.24** Let  $D$  be an  $L$ -domain. Then  $\mathcal{I}(D)$  is an information system with witnesses such that  $D$  and  $\mathcal{L}(\mathcal{I}(D))$  are isomorphic. In addition,

1.  $D$  is algebraic if, and only if, the information system  $\mathcal{I}(D)$  satisfies Condition (ALG).
2.  $D$  is bounded-complete if, and only if, Condition (BC) holds in  $\mathcal{I}(D)$ .

**Proof:** It remains to demonstrate Statements (1) and (2). Because of Theorems 3.18 and 3.19, and as  $D$  and  $\mathcal{L}(\mathcal{I}(D))$  are isomorphic, it suffices to consider just the “only if”-parts, which are obvious, however.

In the remainder of this section we consider two special cases.

**Proposition 3.25** *Let  $T = (\{\Delta\}, \text{CON}_T, \vdash_T, \Delta)$ , where*

$$\text{CON}_T = \{(\Delta, \emptyset), (\Delta, \{\Delta\})\} \text{ and } \vdash_T = \text{CON}_T \times \{\Delta\}.$$

*Then  $T$  is an information system with witnesses satisfying Conditions (ALG) and (BC).  $|T|$  is the one-point domain.*

Let  $(A_1, \text{CON}_1, \vdash_1, \Delta_1)$  and  $(A_2, \text{CON}_2, \vdash_2, \Delta_2)$  be information systems with witnesses, and  $\text{pr}_1$  and  $\text{pr}_2$ , respectively, be the canonical projections of  $A_1 \times A_2$  onto the first and second component. Set  $A_\times = A_1 \times A_2$ ,  $\Delta_\times = (\Delta_1, \Delta_2)$ ,

$$\text{CON}_\times = \{((i, j), X) \in A_\times \times \mathcal{P}_f(A_\times) \mid \text{pr}_1(X) \in \text{CON}_1(i) \wedge \text{pr}_2(X) \in \text{CON}_2(j)\},$$

and for  $((i, j), X) \in \text{CON}_\times$  and  $(a_1, a_2) \in A_\times$  define

$$((i, j), X) \vdash_\times (a_1, a_2) \iff (i, \text{pr}_1(X)) \vdash_1 a_1 \wedge (j, \text{pr}_2(X)) \vdash_2 a_2.$$

Then  $(A_\times, \text{CON}_\times, \vdash_\times, \Delta_\times)$  is an information system with witnesses, the *product* of  $(A_1, \text{CON}_1, \vdash_1, \Delta_1)$  and  $(A_2, \text{CON}_2, \vdash_2, \Delta_2)$ .

**Lemma 3.26** *For  $z \in |A_\times|$  and  $\nu = 1, 2$ , the following two statements hold:*

1.  $\text{pr}_\nu(z) \in |A_\nu|$ .
2.  $z = \text{pr}_1(z) \times \text{pr}_2(z)$ .

**Proof:** (1) Without restriction let  $\nu = 1$ . We only verify Condition 3.6(2), the other two being obvious. Let  $a_1 \in A_1$  and  $(i_1, Y_1) \in \text{CON}_1$  with  $\{i_1\} \cup Y_1 \subseteq \text{pr}_1(z)$  and  $(i_1, Y_1) \vdash_1 a_1$ . Then there are  $i_2 \in A_2$  and  $X \subseteq_{\text{fin}} A_\times$  with  $\text{pr}_1(X) = Y_1$  and  $\{(i_1, i_2)\} \cup X \subseteq z$ . Let  $a_2$  be some element of  $\text{pr}_2(z)$ . Hence, by 3.6(3), there is some  $((j_1, j_2), Z) \in \text{CON}_\times$  with  $\{(j_1, j_2)\} \cup Z \subseteq z$  and  $(j_2, \text{pr}_2(Z)) \vdash_2 a_2$ . Now, by applying Condition (ST), we obtain some  $((k_1, k_2), V) \in \text{CON}_\times$  with  $\{(k_1, k_2)\} \cup V \subseteq z$  so that  $((k_1, k_2), V) \vdash_\times \{(i_1, i_2), (j_1, j_2)\} \cup X \cup Z$ . It follows that  $((k_1, k_2), V) \vdash_\times (a_1, a_2)$ . Thus,  $(a_1, a_2) \in z$ , by 3.6(2), which means that  $a_1 \in \text{pr}_1(z)$ .

(2) is easily shown by applying Conditions 3.6(2, 3).

**Proposition 3.27** *Let  $(A_1, \text{CON}_1, \vdash_1, \Delta_1)$  and  $(A_2, \text{CON}_2, \vdash_2, \Delta_2)$  be information systems with witnesses. Then  $(A_\times, \text{CON}_\times, \vdash_\times, \Delta_\times)$  too is an information system with witnesses and the  $L$ -domains  $|A_\times|$  and  $|A_1| \times |A_2|$  are isomorphic. Moreover,*

1. *If both  $A_1$  and  $A_2$  satisfy Condition (ALG), so does  $A_\times$ .*
2. *If both  $A_1$  and  $A_2$  satisfy Condition (BC), so does  $A_\times$ .*

## 4 Approximable mappings

In the next step we want to turn the collection of information systems with witnesses into a category. The appropriate morphisms are relations similar to entailment relations. In the case of information frames one has to consider families of such relations.

**Definition 4.1** An *approximable mapping*  $H$  between information systems with witnesses  $(A, \text{CON}, \vdash, \Delta)$  and  $(A', \text{CON}', \vdash', \Delta')$ , written  $H: A \Vdash A'$ , is a relation between  $\text{CON}$  and  $A'$  satisfying the following five conditions, for all  $i, j \in A$ ,  $X, X' \subseteq_{\text{fin}} A$ ,  $k \in A'$  and  $Y, F \subseteq_{\text{fin}} A'$  with  $X \in \text{CON}(i)$  and  $Y \in \text{CON}'(k)$ :

1.  $(i, X)H(k, Y) \wedge (k, Y) \vdash' b \Rightarrow (i, X)Hb$
2.  $X' \in \text{CON}(i) \wedge X \subseteq X' \wedge (i, X)Hb \Rightarrow (i, X')Hb$
3.  $(i, X) \vdash X' \wedge (i, X')Hb \Rightarrow (i, X)Hb$
4.  $\{i\} \in \text{CON}(j) \wedge (i, X)Hb \Rightarrow (j, X)Hb$
5.  $(i, X)HF \Rightarrow (\exists(c, U) \in \text{CON})(\exists(e, V) \in \text{CON}')(i, X) \vdash (c, U) \wedge (c, U)H(e, V) \wedge (e, V) \vdash' F$
6.  $(\Delta, \emptyset)H\Delta'$ .

Here,  $(i, X)HY$  means that  $(i, X)Hc$ , for all  $c \in Y$ , and  $(i, X)H(k, Y)$  that  $(i, X)Hk$  as well as  $(i, X)HY$ .

In applications it is sometimes preferable to have Condition (5) split up into two conditions which state interpolation for the domain and the range of the approximable mapping, separately.

**Lemma 4.2** Let  $(A, \text{CON}, \vdash, \Delta)$  and  $(A', \text{CON}', \vdash', \Delta')$  be information systems with witnesses. Then, for any  $H \subseteq \text{CON} \times A'$ ,  $(i, X) \in \text{CON}$ , and  $F \subseteq_{\text{fin}} A'$ , Condition 4.1(5) is equivalent to the following Conditions (1) and (2):

1.  $(i, X)HF \Rightarrow (\exists(c, U) \in \text{CON})(i, X) \vdash (c, U) \wedge (c, U)HF$
2.  $(i, X)HF \Rightarrow (\exists(e, V) \in \text{CON}')(i, X)H(e, V) \wedge (e, V) \vdash' F$ .

**Proof:** The “if”-part follows with 3.4(4) as well as 4.1(4, 3), and 4.1(1), respectively. The “only if”-part is obvious.

Similar to Lemma 3.5 a strengthening of Axiom 4.1(3) can be derived. It reverses the implication in Lemma 4.2(1).

**Lemma 4.3** Let  $H$  be an approximable mapping between information systems  $A$  and  $A'$  with witnesses. Then for all  $(i, X), (j, Y) \in \text{CON}$  and  $b \in A'$ ,

$$(i, X) \vdash (j, Y) \wedge (j, Y)Hb \Rightarrow (i, X)Hb.$$

As has already been mentioned, entailment relations are special approximable mappings. For  $(i, X) \in \text{CON}$  and  $a \in A$ , set  $(i, X) \text{Id}_A a$  if  $(i, X) \vdash a$ . Then  $\text{Id}: A \Vdash A$  such that for all  $H: A \Vdash A'$ ,  $H \circ \text{Id}_{A'} = H = \text{Id}_A \circ H$ , where for approximable mappings  $H: A \Vdash A'$  and  $G: A' \Vdash A''$  their composition  $H \circ G: A \Vdash A''$  is defined by

$$(i, X)(H \circ G)c \iff (\exists(j, Y) \in \text{CON}') (i, X)H(j, Y) \wedge (j, Y)Gc.$$

Let **ISW** be the category of information systems with witnesses and approximable mappings and **aISW**, **bcISW**, and **abcISW**, respectively, be the full subcategories of information systems with witnesses that satisfy Condition (ALG), Condition (BC), or both of them.

**Proposition 4.4** *The one-point information system  $T$  with witnesses is a terminal object in **ISW**.*

**Proof:** Let  $(A', \text{CON}', \vdash', \Delta')$  be an information system with witnesses and  $H = \text{CON}' \times \{\Delta\}$ . It suffices to show that  $H: A' \Vdash T$ . We only verify Condition 4.1(5), the others being obvious.

Let to this end  $(i, X) \in \text{CON}'$  with  $(i, X)HF$ , where  $F = \emptyset$  or  $F = \{\Delta\}$ . Then  $(i, X) \vdash' (\Delta', \emptyset)$ ,  $(\Delta', \emptyset)H(\Delta, \emptyset)$ , and  $(\Delta, \emptyset) \vdash_T F$ .

As  $T$  satisfies both (ALG) and (BC), it is of course also terminal in **aISW**, **bcISW** and **abcISW**.

For two information systems  $(A_1, \text{CON}_1, \vdash_1, \Delta_1)$  and  $(A_2, \text{CON}_2, \vdash_2, \Delta_2)$  with witnesses define the relations  $\text{Pr}_\nu \subseteq \text{CON}_\times \times A_\nu$ , for  $\nu = 1, 2$ , by

$$((i_1, i_2), X) \text{Pr}_\nu a_\nu \iff (i_\nu, \text{pr}_\nu(X)) \vdash_\nu a_\nu.$$

**Lemma 4.5** *For  $\nu = 1, 2$ ,  $\text{Pr}_\nu: A_\times \Vdash A_\nu$ .*

**Proof:** Again, we verify only Condition 4.1(5). Let  $((i_1, i_2), X) \in \text{CON}_\times$  and  $F \subseteq_{\text{fin}} A_\nu$  with  $((i_1, i_2), X) \text{Pr}_\nu F$ . Then  $(i_\nu, \text{pr}_\nu(X)) \vdash_\nu F$ . Hence, there are  $(j_\nu, Y_\nu), (k_\nu, Z_\nu) \in \text{CON}_\nu$  so that  $(i_\nu, \text{pr}_\nu(X)) \vdash_\nu (j_\nu, Y_\nu) \vdash_\nu (k_\nu, Z_\nu) \vdash_\nu F$ . Set  $Y = Y_1 \times \{\Delta_2\}$  and  $j_2 = \Delta_2$ , if  $\nu = 1$ , and  $Y = \{\Delta_1\} \times Y_2$  as well as  $j_1 = \Delta_1$ , otherwise. It follows that  $((i_1, i_2), X) \vdash_\times ((j_1, j_2), Y) \text{Pr}_\nu(k_\nu, Z_\nu) \vdash_\nu F$ .

**Proposition 4.6** *For information systems  $A_1$  and  $A_2$  with witnesses,  $(A_\times, \text{Pr}_1, \text{Pr}_2)$  is their categorical product.*

Note that for approximable mappings  $H_1: A \Vdash A_1$  and  $H_2: A \Vdash A_2$  the mediating morphism  $\langle H_1, H_2 \rangle: A \Vdash A_\times$  is given by

$$(i, X)(H_1, H_2)(a_1, a_2) \iff (i, X)H_1a_1 \wedge (i, X)H_2a_2.$$

As we have already seen, there is a close connection between information systems with witnesses and L-domains. It can be extended to the corresponding morphisms, i.e. approximable mappings and Scott continuous functions, so that we obtain an equivalence between **ISW** and **L**.

Let  $(A, \text{CON}, \vdash, \Delta)$  and  $(A', \text{CON}', \vdash', \Delta')$  be information systems with witnesses and  $H: A \Vdash A'$ .

**Lemma 4.7** *For  $x \in |A|$ ,*

$$\{a \in A' \mid (\exists(i, X) \in \text{CON})\{i\} \cup X \subseteq x \wedge (i, X)Ha\} \in |A'|.$$

**Proof:** For the verification of Condition 3.6(2) set

$$y = \{a \in A' \mid (\exists(i, X) \in \text{CON})\{i\} \cup X \subseteq x \wedge (i, X)Ha\}$$

and let  $a \in A'$ . Moreover, let  $(j, Y) \in \text{CON}'$  such that  $\{j\} \cup Y \subseteq y$  and  $(j, Y) \vdash' a$ . Then there exists  $(i, X) \in \text{CON}$  with  $\{i\} \cup X \subseteq x$  and  $(i, X)Hj$ . In addition, for every  $e \in Y$ , there is  $(b_e, Z_e) \in \text{CON}$  so that  $\{b_e\} \cup Z_e \subseteq x$  and  $(b_e, Z_e)He$ . Set  $F = \{i\} \cup X \cup \bigcup \{Z_e \mid e \in Y\} \cup \bigcup \{b_e \mid e \in Y\}$ . By (ST) there is thus some  $(k, U) \in \text{CON}$  with  $\{k\} \cup U \subseteq x$  and  $(k, U) \vdash F$ . It follows that

$$(k, U) \vdash (i, X) \text{ and } (i, X)Hj,$$

whence, by Lemma 4.3, we obtain that  $(k, U)Hj$ . Similarly, for every  $e \in Y$ , we have  $(k, U) \vdash (b_e, Z_e)$  and  $(b_e, Z_e)He$ , and hence that  $(k, U)He$ . So, we gain that  $(k, U)H(j, Y)$ . Since  $(j, Y) \vdash' a$ , an application of 4.1(1) yields  $(k, U)Ha$ . Thus,  $a \in y$ .

It remains to verify Condition (ST). Let to this end,  $F \subseteq_{\text{fin}} y$ . Then, for any  $e \in F$ , there is some  $(i_e, X_e) \in \text{CON}$  with  $\{i_e\} \cup X_e \subseteq x$  and  $(i_e, X_e)He$ . Set  $K = \{i_e \mid e \in F\} \cup \bigcup \{X_e \mid e \in F\}$ . Then  $K \subseteq_{\text{fin}} x$ . Thus, it follows with Condition (ST) that there is  $(j, U) \in \text{CON}$  so that  $\{j\} \cup U \subseteq x$  and  $(j, U) \vdash K$ . In particular, we have that  $(j, U) \vdash (i_e, X_e)$ , for every  $e \in F$ . With Lemma 4.3 we therefore obtain that  $(j, U)HF$ . Because of Lemma 4.2(2) there is now some  $(k, V) \in \text{CON}'$  with  $(j, U)H(k, V)$  and  $(k, V) \vdash F$ . It remains to show that  $\{k\} \cup V \subseteq y$ , which, however, is a consequence of  $(j, U)H(k, V)$ .

This allows us to define a function  $\mathcal{L}(H): \mathcal{L}(A) \rightarrow \mathcal{L}(A')$  by

$$\mathcal{L}(H)(x) = \{a \in A' \mid (\exists(i, X) \in \text{CON})\{i\} \cup X \subseteq x \wedge (i, X)Ha\}.$$

**Lemma 4.8**  $\mathcal{L}(H)$  is Scott continuous.

**Proof:** Obviously,  $\mathcal{L}(H)$  is monotone. Let  $\mathcal{S}$  be a directed subset of  $|A|$ . Then it remains to show that  $\mathcal{L}(H)(\bigcup \mathcal{S}) \subseteq \bigcup \mathcal{L}(H)(\mathcal{S})$ , the converse inclusion being a consequence of monotonicity.

Let  $b \in \mathcal{L}(H)(\bigcup \mathcal{S})$ . Then there is some  $(i, X) \in \text{CON}$  with  $\{i\} \cup X \subseteq \bigcup \mathcal{S}$  and  $(i, X)Hb$ . Since  $\mathcal{S}$  is directed and  $X$  finite, it follows that  $\{i\} \cup X \subseteq x$ , for some  $x \in \mathcal{S}$ . Thus,  $b \in \mathcal{L}(H)(x)$ .

**Lemma 4.9**  $\mathcal{L}: \mathbf{IWS} \rightarrow \mathbf{L}$  is a functor.

**Proof:** Because of Conditions 3.6(2, 3) we have that

$$\mathcal{L}(\text{Id}_A)(x) = \{a \in A \mid (\exists(i, X) \in \text{CON})\{i\} \cup X \subseteq x \wedge (i, X) \vdash a\} = x.$$

Thus,  $\mathcal{L}(\text{Id}_A) = \text{id}_{|A|}$ , where  $\text{id}_{|A|}$  is the identity function on  $|A|$ . It remains to show functoriality.

Let to this end  $(A'', \text{CON}'', \vdash'', \Delta'')$  be a further information system with witnesses, and  $H: A \Vdash A'$  as well as  $G: A' \Vdash A''$ . Moreover, let  $c \in \mathcal{L}(G)(\mathcal{L}(H)(x))$ , for  $x \in |A|$ . Then there is some  $(j, Y) \in \text{CON}'$  with  $\{j\} \cup Y \subseteq \mathcal{L}(H)(x)$  and  $(j, Y)Gc$ . Let  $F = \{j\} \cup Y$ . Then it follows that for every  $b \in F$  there is some  $(i_b, X_b) \in \text{CON}$  so that  $\{i_b\} \cup X_b \subseteq x$  and  $(i_b, X_b)Hb$ . Set  $K = \{i_b \mid b \in F\} \cup \bigcup \{X_b \mid b \in F\}$ . Then  $K \subseteq_{\text{fin}} x$ . Hence, there is some  $(k, Z) \in \text{CON}$  with  $\{k\} \cup Z \subseteq x$  and  $(k, Z) \vdash K$ . Consequently,  $(k, Z)H(j, Y)$ . This shows that  $c \in \mathcal{L}(H \circ G)(x)$ .

Now, conversely, let  $c \in \mathcal{L}(H \circ G)(x)$ . Then there is some  $(i, X) \in \text{CON}$  so that  $\{i\} \cup X \subseteq x$  and  $(i, X)(H \circ G)c$ . It follows that there is also some  $(j, Y) \in \text{CON}'$  with  $(i, X)H(j, Y)$  and  $(j, Y)Gc$ . Thus,  $\{j\} \cup Y \subseteq \mathcal{L}(H)(x)$ . So, we have that  $c \in \mathcal{L}(G)(\mathcal{L}(H)(x))$ .

Let us next consider the converse situation in which we went from L-domains to information systems with witnesses. As we will see, every Scott continuous function  $f: D \rightarrow D'$  between L-domains  $D$  and  $D'$  defines an approximable mapping  $\mathcal{I}(f): \mathcal{I}(D) \Vdash \mathcal{I}(D')$ .

Let  $D$  and  $D'$ , respectively, have bases  $B$  and  $B'$ . Then, for  $i \in B$ ,  $X \subseteq_{\text{fin}} \downarrow i \cap B$ , and  $a \in B'$ , set

$$(i, X)\mathcal{I}(f)a \iff a \ll' f(\bigsqcup^i X).$$

**Lemma 4.10**  $\mathcal{I}(f): \mathcal{I}(D) \Vdash \mathcal{I}(D')$ .

**Proof:** Let  $i \in B$ ,  $X \subseteq_{\text{fin}} \downarrow i \cap B$ ,  $k, b \in B'$ , and  $Y, S \subseteq_{\text{fin}} \downarrow k \cap B'$ . We have to verify Conditions 4.1(1-6).

(1) Assume that  $(i, X)\mathcal{I}(f)(k, Y)$  and  $(k, Y) \vdash' b$ . Then we have that  $b \ll' \bigsqcup^k Y \ll' f(\bigsqcup^i X)$ , where in the last case we had to apply the interpolation law first. It follows that  $b \ll' f(\bigsqcup^i X)$ . Thus,  $(i, X)\mathcal{I}(f)b$ .

Condition (3) follows in a similar way, and Conditions (2), (4) and (6) are obvious. We consider only Condition (5).

Assume that  $(i, X)\mathcal{I}(f)S$ . Then  $S \ll' f(\bigsqcup^i X)$ . Because of the interpolation law there is some  $e \in B'$  with  $S \ll' e \ll' f(\bigsqcup^i X)$ . As  $f$  is Scott continuous, we have that  $f(\bigsqcup^i X) = \bigsqcup f(\{a \in B \mid a \ll \bigsqcup^i X\})$ . It follows that there is some  $c \in B$  with  $c \ll \bigsqcup^i X$  so that  $e \ll' f(c)$ . Thus, we obtain that  $(i, X) \vdash (c, \{c\})$ ,  $(c, \{c\})\mathcal{I}(f)(e, \{e\})$  and  $(e, \{e\}) \vdash' S$ .

**Lemma 4.11**  $\mathcal{I}: \mathbf{L} \rightarrow \mathbf{ISW}$  is a functor.

Functoriality follows from Scott continuity. The other property is obvious.

As we have seen in the preceding section, up to isomorphism every L-domain is generated by an information system with witnesses. Let  $A$  and  $A'$  be such information systems and  $f: |A| \rightarrow |A'|$  Scott continuous. If we now construct the information systems corresponding to the domains  $|A|$  and  $|A'|$  as well as the approximable mapping corresponding to  $f$  in the above way and consider the domains and the function generated by these, we will not come back to  $f$ . The state sets involved will be one level higher up in the power set hierarchy. This can be avoided, however. For  $(i, X) \in \text{CON}$  and  $a \in A'$ , define

$$(i, X)H^f a \iff a \in f([X]_i).$$

**Lemma 4.12**  $H^f: A \Vdash A'$ .

**Proof:** We have to verify the conditions in Definition 4.1.

(1) Assume that  $(i, X)H^f(k, Y)$  and  $(k, Y) \vdash' b$ . Then  $\{k\} \cup Y \subseteq_{\text{fin}} f([X]_i)$ . Since  $f([X]_i)$  is a state, it follows with ST that  $(j, Z) \vdash' (k, Y)$ , for some  $(j, Z) \in \text{CON}'$  with  $\{j\} \cup Z \subseteq f([X]_i)$ . Then  $(j, Z) \vdash' b$  and hence  $b \in f([X]_i)$ , by 3.6(2).

(2, 3) are obvious by the monotonicity of  $f$ . (4) is obvious as well, as is (6), since  $\Delta'$  is contained in any state of  $A'$ , by 3.6(2).

It remains to verify (5). Let  $(i, X)H^f F$ . Then, by Condition (ST), there is some  $(j, Y) \in \text{CON}'$  with  $\{j\} \cup Y \subseteq f([X]_i)$  and  $(j, Y) \vdash' F$ . Now, Condition 3.4(10) provides us with some  $(e, V) \in \text{CON}'$  so that  $(j, Y) \vdash' (e, V)$  and  $(e, V) \vdash' F$ . It follows with 3.6(2) that  $\{e\} \cup V \subseteq f([X]_i)$ . By Lemma 3.9(2) and the Scott continuity of  $f$  we therefore obtain that there is some  $(k, Z) \in \text{CON}$  with  $(i, X) \vdash (k, Z)$  and  $\{e\} \cup V \subseteq f([Z]_k)$ . Applying 3.4(10) again supplies us with some  $(c, U) \in \text{CON}$  such that  $(i, X) \vdash (c, U)$  and  $(c, U) \vdash (k, Z)$ . It ensues that  $\{e\} \cup V \subseteq f([Z]_k) \subseteq f([U]_c)$ . Altogether we thus have that  $(i, X) \vdash (c, U)$ ,  $(c, U)H^f(e, V)$  and  $(e, V) \vdash' F$ .



**Lemma 4.13**  $\mathcal{L}(H^f) = f$ .

**Proof:** Let  $x \in |A|$  and  $b \in A'$ . Then we have

$$\begin{aligned}
b \in \mathcal{L}(H^f)(x) &\iff (\exists(i, X) \in \text{CON})\{i\} \cup X \subseteq x \wedge (i, X)H^f b \\
&\iff (\exists(i, X) \in \text{CON})\{i\} \cup X \subseteq x \wedge b \in f([X]_i) \\
&\iff b \in \bigcup \{f([X]_i) \mid (i, X) \in \text{CON} \wedge \{i\} \cup X \subseteq x\} \\
&\iff b \in f(\bigcup \{[X]_i \mid (i, X) \in \text{CON} \wedge \{i\} \cup X \subseteq x\}) \\
&\iff b \in f(x).
\end{aligned}$$

Now, conversely, let  $H: A \Vdash A'$ . Then  $\mathcal{L}(H): |A| \rightarrow |A'|$ , by Lemma 4.8.

**Lemma 4.14**  $H^{\mathcal{L}(H)} = H$ .

**Proof:** For  $(i, X) \in \text{CON}$  and  $b \in A'$ , we have that

$$\begin{aligned}
(i, X)H^{\mathcal{L}(H)}b &\iff b \in \mathcal{L}(H)([X]_i) \\
&\iff (\exists(j, Y) \in \text{CON})(i, X) \vdash (j, Y) \wedge (j, Y)Hb \\
&\iff (i, X)Hb.
\end{aligned}$$

The last equivalence follows with Lemmas 4.3 and 4.2(1), respectively.

As a consequence of the last two lemmas the functor  $\mathcal{L}: \mathbf{ISW} \rightarrow \mathbf{L}$  is full and faithful. Moreover, by Theorem 3.24, we have that any domain  $D$  in  $\mathbf{L}$  is isomorphic to  $\mathcal{L}(A)$ , for some information system  $A$  in  $\mathbf{ISW}$ , namely  $\mathcal{I}(D)$ . With [13, p. 93, Theorem 1] it thus follows that  $\mathcal{L}$  is an equivalence.

**Theorem 4.15** *The category  $\mathbf{ISW}$  of information systems with witnesses and approximable mappings is equivalent to the category  $\mathbf{L}$  of L-domains and Scott continuous functions.*

**Corollary 4.16** *The categories  $\mathbf{aISW}$ ,  $\mathbf{bcISW}$  and  $\mathbf{abcISW}$ , respectively, of information systems with witnesses satisfying Conditions (ALG), (BC), or both of them, and approximable mappings are equivalent to the categories  $\mathbf{aL}$ ,  $\mathbf{BC}$  and  $\mathbf{aBC}$  of algebraic L-domain, bc-domains and algebraic bc-domains with Scott continuous functions.*

## 5 Other kinds of information systems

In this section we will see how some classical types of information systems studied in the literature can be considered as information systems with witnesses in which the witnesses are ignored.

Scott [14] introduced his information systems as a logic-based introduction to countably based algebraic bc-domains. In order to capture the more general continuous bc-domains, Hoofman [9] extended Scott's approach and introduced continuous information systems. Moreover, he showed that Scott's information systems or, more exactly, its slight modification introduced by Larsen and Winskel [12] are a special case of his information systems.

## 5.1 Continuous information systems

**Definition 5.1** Let  $A$  be a set,  $\text{Con} \subseteq \mathcal{P}_f(A)$ , and  $\vdash \subseteq \text{Con} \times A$ .  $(A, \text{Con}, \vdash)$  is a *continuous information system* if the following conditions hold, for all  $a \in A$  and  $X, Y \subseteq_{\text{fin}} A$ :

1.  $\emptyset \in \text{Con}$
2.  $Y \subseteq X \wedge X \in \text{Con} \Rightarrow Y \in \text{Con}$
3.  $\{a\} \in \text{Con}$
4.  $X \vdash Y \Rightarrow Y \in \text{Con}$
5.  $(X, Y \in \text{Con} \wedge X \subseteq Y \wedge X \vdash a) \Rightarrow Y \vdash a$
6.  $(\exists Z \in \text{Con})[X \vdash Z \wedge Z \vdash a] \Leftrightarrow X \vdash a$ .

Set  $\text{CON} = A \times \text{Con}$  and for  $(i, X) \in \text{CON}$  define

$$(i, X) \models a \iff X \vdash a.$$

**Proposition 5.2** Let  $(A, \text{Con}, \vdash)$  be a continuous information system. Then  $(A \cup \{\emptyset\}, \text{CON}, \models, \emptyset)$  is an information system with witnesses that satisfies Condition (BC).

**Proof:** Condition (BC) is obvious and Conditions 3.4(1-9) are easily verified and Condition 3.4(10) is a consequence of [9, Theorem 20].

As we shall see next, there is a canonical way to pass from an information system with witnesses satisfying Condition (BC) to a continuous information system such that both generate the same domain.

Let  $(A, \text{CON}, \vdash, \Delta)$  be an information system with witnesses so that Requirement (BC) holds. Set  $\text{Con} = \bigcup \{ \text{CON}(i) \mid i \in A \}$  and for  $a \in A$  and  $X \subseteq_{\text{fin}} A$  define

$$X \Vdash a \iff (\exists i \in A) X \in \text{CON}(i) \wedge (i, X) \vdash a.$$

**Lemma 5.3**  $\mathcal{C}(A) = (A, \text{Con}, \Vdash)$  is a continuous information system.

**Proof:** We have to verify the conditions in Definition 5.1.

(1) By Condition 3.4(1) we have that  $\{\Delta\} \in \text{CON}(\Delta)$ . Because of 3.4(2) it follows that  $\emptyset \in \text{CON}(\Delta)$  and hence that  $\emptyset \in \text{Con}$ .

(2) Let  $Y \in \text{Con}$ . Then there is some  $i \in A$  with  $Y \in \text{CON}(i)$ . With Condition 3.4(2) it follows for  $X \subseteq Y$  that  $X \in \text{CON}(i)$  as well. Thus,  $X \in \text{Con}$ .

(3) Since  $\{a\} \in \text{CON}(a)$  by Axiom 3.4(1), we have that  $\{a\} \in \text{Con}$ , for every  $a \in A$ .

(4) Let  $X \in \text{Con}$  and  $Y \subseteq_{\text{fin}} A$  with  $X \Vdash Y$ . Then there is some  $i_b \in A$  with  $X \in \text{CON}(i_b)$  and  $(i_b, X) \vdash b$ , for each  $b \in Y$ . Thus,  $X \in \bigcap \{ \text{CON}(i_b) \mid b \in Y \}$ . Because of Condition (BC) we moreover have that for all  $c, d \in Y$ ,

$$(i_c, X) \vdash b \iff (i_d, X) \vdash b.$$

If  $Y$  is empty, we already know that  $Y \in \text{Con}$ . Otherwise, let  $c \in Y$ . Then,  $(i_c, X) \vdash b$ , for all  $b \in Y$ , that is,  $(i_c, X) \vdash Y$ . With 3.4(4) it follows that  $Y \in \text{CON}(i_c)$  which shows that  $Y \in \text{Con}$ .

(5) Let  $a \in A$  and  $X, Y \in \text{Con}$  such that  $X \Vdash a$  and  $X \subseteq Y$ . We must show that  $Y \Vdash a$ . As a consequence of our assumption there are  $i, j \in A$  so that  $X \in \text{CON}(i)$ ,  $(i, X) \vdash a$  and  $Y \in \text{CON}(j)$ . Since  $X \subseteq Y$ , it follows with Axiom 3.4(2) that  $X \in \text{CON}(j)$  as well. Thus,  $X \in \text{CON}(i) \cap \text{CON}(j)$ . Because of (BC) we therefore have that  $(i, X) \vdash a$ , exactly if  $(j, X) \vdash a$ . So,  $X, Y \in \text{CON}(j)$ ,  $(j, X) \vdash a$  and  $X \subseteq Y$  which implies that  $(j, Y) \vdash a$ , by Axiom 3.4(5). Hence,  $Y \Vdash a$ .

(6) Assume that  $X \Vdash Y$  and  $Y \Vdash a$ . We first need to show that  $X \Vdash a$ . Since  $X \Vdash Y$ , there is some  $i_b \in A$ , for each  $b \in Y$ , such that  $X \in \text{CON}(i_b)$  and  $(i_b, X) \vdash b$ . Moreover, as  $Y \Vdash a$ , there is some  $j \in A$  with  $Y \in \text{CON}(j)$  and  $(j, Y) \vdash a$ . If  $Y$  is empty, we have that  $\emptyset \Vdash a$  and therefore, by what has just been shown, that  $X \Vdash a$ . Otherwise, there is some  $c \in Y$ . Because  $X \in \bigcap \{ \text{CON}(i_b) \mid b \in Y \}$ , it follows with (BC) that for all  $b \in Y$ ,  $(i_b, X) \vdash b$ , exactly if  $(i_c, X) \vdash b$ . Thus, we have that  $(i_c, X) \vdash Y$ , which by Axiom 3.4(4) implies that  $Y \in \text{CON}(i_c)$ . This shows that  $Y \in \text{CON}(j) \cap \text{CON}(i_c)$ . With (BC) we hence obtain that  $(i_c, Y) \vdash a$ , exactly if  $(j, Y) \vdash a$ . Altogether we thus have that  $(i_c, X) \vdash Y$  and  $(i_c, Y) \vdash a$ . Therefore,  $(i_c, X) \vdash a$ , because of 3.4(6), which means that  $X \Vdash a$ .

The converse implication is an easy consequence of Axiom 3.4(10).

Let  $(A, \text{Con}, \vdash)$  be a continuous information system. A subset  $x$  of  $A$  is a *point*, if the following three requirements hold:

1.  $(\forall F \subseteq_{\text{fin}} x) F \in \text{Con}$
2.  $(\forall X \subseteq_{\text{fin}} x)(\forall a \in A)[X \in \text{Con} \wedge X \Vdash a \Rightarrow a \in x]$
3.  $(\forall a \in x)(\exists X \subseteq_{\text{fin}} x) X \in \text{Con} \wedge X \Vdash a$ .

The collection of points is a continuous bc-domain with respect to set inclusion [9] which we also denote by  $\mathcal{L}(A)$ .

Let  $(A, \text{CON}, \vdash, \Delta)$  be an information system with witnesses such that Condition (BC) holds. Then every state of  $(A, \text{CON}, \vdash, \Delta)$  is a point of  $\mathcal{L}(A)$ , and conversely. So, both systems generate the same domain.

**Proposition 5.4** *Let  $(A, \text{CON}, \vdash, \Delta)$  be an information system with witnesses such that Condition (BC) holds. Then  $\mathcal{L}(A)$  is a continuous information system such that  $\mathcal{L}(A) = \mathcal{L}(\mathcal{C}(A))$ .*

## 5.2 Algebraic information systems

Next, we consider the algebraic case.

**Definition 5.5** Let  $A$  be a set,  $\Delta \in A$ ,  $\text{Con} \subseteq \mathcal{P}_f(A)$ , and  $\vdash \subseteq \text{Con} \times A$ .  $(A, \text{Con}, \vdash, \Delta)$  is an *algebraic information system* if for all  $a \in A$  and  $X, Y \subseteq_{\text{fin}} A$  the following requirements are satisfied:

1.  $Y \subseteq X \wedge X \in \text{Con} \Rightarrow Y \in \text{Con}$
2.  $\{a\} \in \text{Con}$
3.  $X \vdash a \Rightarrow X \cup \{a\} \in \text{Con}$
4.  $X \vdash \Delta$
5.  $(X, Y \in \text{Con} \wedge X \vdash Y \wedge X \vdash a) \Rightarrow Y \vdash a$

6.  $a \in X \Rightarrow X \vdash a$ .

Every algebraic information system is a continuous information system. Let  $\text{CON}$  and  $\models$  be defined as in Proposition 5.2.

**Proposition 5.6** *Let  $(A, \text{Con}, \vdash, \Delta)$  be an algebraic information system. Then  $(A, \text{CON}, \models, \Delta)$  is an information system with witnesses so that Requirements (BC) and (ALG) hold.*

**Proof:** Conditions 3.4(1-3, 6-9) and (BC) are obvious, Condition 3.4(5) is a consequence of Requirements 5.5(5, 6), and 3.4(4) follows with 3.4(5) and 5.5(3). With Requirement 5.5(6) we also obtain that Conditions (SALG) and hence 3.4(10) hold.

For the converse construction we require that the following strengthening (ALG+) of Condition (ALG) holds:

$$(i, X) \vdash F \Rightarrow (\exists j \in A)(i, X) \vdash j \wedge (j, \{j\}) \vdash j \wedge (j, \{j\}) \vdash F. \quad (\text{ALG+})$$

Note that Condition (ALG+) is satisfied by all information systems with witnesses  $\mathcal{I}(D)$  generated by algebraic L-domains  $D$ .

We say that  $j \in A$  is *reflexive* if  $(j, \{j\}) \vdash j$ , and denote the subset of all such elements in  $A$  by  $A_{\text{refl}}$ .

**Lemma 5.7** *Let  $(j, V) \in \text{CON}$  with  $\{j\} \cup V \subseteq A_{\text{refl}}$ . Then  $(j, V)$  is reflexive.*

**Proof:** Let  $a \in \{j\} \cup V$ . Since  $(j, V) \in \text{CON}$ , it follows with 3.4(2) that  $\{a\} \in \text{CON}(j)$ . According to our assumption we have that  $(a, \{a\}) \vdash a$ . With 3.4(8) we therefore obtain that also  $(j, \{a\}) \vdash a$ . Hence,  $(j, V) \vdash a$ , because of 3.4(5). This shows that  $(j, V) \vdash (j, V)$ .

Let  $(A, \text{CON}, \vdash, \Delta)$  be an information system with witnesses such that both Conditions, (BC) and (ALG+), hold. Set

$$\text{Con}_{\text{refl}} = \{ X \subseteq_{\text{fin}} A_{\text{refl}} \mid (\exists i \in A_{\text{refl}})(i, X) \in \text{CON} \}$$

and for  $a \in A_{\text{refl}}$  as well as  $X \subseteq_{\text{fin}} A_{\text{refl}}$ ,

$$X \vdash_{\text{refl}} a \iff (\exists i \in A_{\text{refl}})(i, X) \in \text{CON} \wedge (i, X) \vdash a.$$

Note that  $\Delta \in A_{\text{refl}}$  because of Axioms 3.4(3-5).

**Lemma 5.8** *Let  $(A, \text{CON}, \vdash, \Delta)$  be an information system with witnesses such that Conditions (BC) and (ALG+) hold. Then  $\mathcal{R}(A) = (A_{\text{refl}}, \text{Con}_{\text{refl}}, \vdash_{\text{refl}}, \Delta)$  is an algebraic information system.*

**Proof:** We have to verify the requirements in Definition 5.5. For 5.5(1, 2, 5) the proof proceeds as in Lemma 5.3.

(3) Assume that  $X \vdash_{\text{refl}} a$ . Then there is some  $i \in A_{\text{refl}}$  such that  $(i, X) \in \text{CON}$  and  $(i, X) \vdash a$ . As  $(i, X)$  is reflexive, by Lemma 5.7, we have that  $(i, X) \vdash \{i, a\} \cup X$ . With Axiom 3.4(4) it follows that  $X \cup \{a\} \in \text{CON}(i)$ . Therefore,  $(i, X \cup \{a\})$  is reflexive, again by Lemma 5.7. Thus,  $X \cup \{a\} \in \text{Con}$ .

(4) Since  $X \in \text{Con}$ , there is some  $i \in A_{\text{refl}}$  so that  $(i, X) \in \text{CON}$ . By Axiom 3.4(3) we have in addition that  $(i, \emptyset) \vdash \Delta$ . With 3.4(5) therefore obtain that  $(i, X) \vdash \Delta$ , which means that  $X \vdash_{\text{refl}} \Delta$ .

(6) Suppose that  $a \in X$ , where  $X \in \text{Con}_{\text{refl}}$ . Then  $(i, X) \in \text{CON}$ , for some  $i \in A_{\text{refl}}$ . With Lemma 5.7 it follows that  $(i, X)$  is reflexive. Hence,  $(i, X) \vdash a$ , that is,  $X \vdash_{\text{refl}} a$ .

For algebraic information systems the definition of a point can be simplified. Let  $(A, \text{Con}, \vdash, \Delta)$  be an algebraic information system. A subset  $x$  of  $A$  is a *point*, if the following two requirements hold:

1.  $(\forall F \subseteq_{\text{fin}} x) F \in \text{Con}$
2.  $(\forall X \subseteq_{\text{fin}} x)(\forall a \in A)[X \in \text{Con} \wedge X \vdash a \Rightarrow a \in x]$ .

The collection of points is an algebraic bc-domain with respect to set inclusion [14] and  $\{ \{ a \in A \mid \{ i \} \vdash a \} \mid i \in A \}$  is its canonical basis. We denote this domain by  $\mathcal{L}(A)$  as well.

Let  $(A, \text{CON}, \vdash, \Delta)$  be an information system with witnesses such that Conditions (BC) and (ALG+) hold. Then  $\mathcal{L}(A)$  is an algebraic bc-domain with basis  $\{ [\{ i \}]_i \mid i \in A_{\text{refl}} \}$ . It follows that the two information systems  $A$  and  $\mathcal{R}(A)$  generate isomorphic domains.

**Proposition 5.9** *Let  $(A, \text{CON}, \vdash, \Delta)$  be an information system with witnesses such that Conditions (BC) and (ALG+) hold. Then  $\mathcal{R}(A)$  is an algebraic information system such that  $\mathcal{L}(\mathcal{R}(A))$  and  $\mathcal{L}(A)$  are isomorphic domains.*

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